PERTURBATION OF THE FRONT OF A FLAT PLASTIC WAVE

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We consider the system of equations of viscoelasticity in the following form [1]:

$$\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha} - \frac{1}{\rho}\frac{\partial \sigma_{ik}}{\partial x_k} = 0,\ i = 1, 2, 3,$$
$$\frac{\partial g_{ij}}{\partial t} + u_\alpha \frac{\partial g_{ij}}{\partial x_\alpha} - g_{i\alpha}\frac{\partial u_\alpha}{\partial x_j} + g_{j\alpha}\frac{\partial u_\alpha}{\partial x_i} = \varphi_{ij},\ i, j = 1, 2, 3, \quad (1)$$
$$\frac{\partial S}{\partial t} + u_\alpha \frac{\partial S}{\partial x_\alpha} = \varkappa,$$

where x_1, x_2, x_3 are Euler (Cartesian) coordinates; u_i are the components of the vector of the velocity; g_{ij} are the components of the tensor of the deformations; S is the entropy; $\sigma_{ij} = -2\rho g_{i\alpha}\partial E/g\partial_{\alpha j}$ is the stress tensor; $E = E(k_1, k_2, k_3, S)$ is the density of the internal energy; $k_i = \sqrt{g_i}$, g_i are the principal values of the tensor g_{ij}; $\rho = \rho_0\sqrt{\det \|g_{ij}\|}$ is the density of the medium.

The right-hand parts φ_{ij} in the equations for g_{ij} take account of the inelastic deformations and, in the given model, represent the Maxwell relaxation terms; κ takes account of the rise in the entropy with inelastic deformations and is expressed in terms of φ_{ij} from the law of conservation of energy $\kappa = -E_{g_{ij}}\varphi_{ij}/E_S$. We choose φ_{ij} in the following manner: $\varphi_{ii} = \frac{2}{\tau} g_{ii}\left(h_i - \frac{h_1 + h_2 + h_3}{3}\right)$, $h_i = \ln k_i$, $\varphi_{ij} = -(1/\tau)g_{ij}$ $(i \neq j)$, where $\tau(k_1, k_2, k_3, S)$ is the characteristic relaxation time of the tangential stresses. For τ, we shall use interpolation formulas given in [2], which represent a dependence of the form $\tau = \tau(\sigma, T)$, where σ is the intensity of the tangential stresses; T is the temperature. Interpolation formulas for the internal energy are given in [3].

A flat, stationary wave is a solution of system (1) of the form $u_1(x_1)$, $u_2 = u_3 = 0$, $g_{11}(x_1)$, $g_{22}(x_1) = g_{33}(x_1)$, $g_{ij} = 0$ $(i \neq j)$, $S(x_1)$ (in what follows, we denote $x = x_1$, $y = x_2$, $z = x_3$, $u = u_1$, $v = u_2$, $w = u_3$). This solution satisfies a one-dimensional system of equations, which can be obtained from (1)

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0,\ \frac{\partial \rho u}{\partial t} + \frac{\partial(\rho u^2 - \sigma_1)}{\partial x} = 0,$$
$$\frac{\partial \rho\left(E + \frac{u^2}{2}\right)}{\partial t} + \frac{\partial\left[\rho u\left(E + \frac{u^2}{2}\right) - u\sigma_1\right]}{\partial x} = 0,$$
$$\frac{\partial h_2}{\partial t} + u\frac{\partial h_2}{\partial x} = \frac{h_1 - h_2}{3\tau},$$

where $h_i = \ln g_i$; $\rho = \rho_0 \exp(-h_1 - h_2 - h_3)$; $E = E(h_1, h_2, h_3, S)$; $\sigma_i = \rho E_{h_i}$.

Stationary waves satisfy the system

$$[\rho u] = 0,\ [\rho u^2 - \sigma_1] = 0,$$
$$\left[\rho u\left(E + \frac{u^2}{2}\right) - u\sigma_1\right] = 0,\ u\frac{dh_2}{dx} = \frac{h_1 - h_2}{3\tau} \quad (2)$$

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and the following conditions:

with $x \to -\infty$

$$u \to u_0,\ \rho \to \rho_0,\ h_2 \to 0,\ S \to 0,$$
$$\frac{du}{dx} \to 0,\ \frac{d\rho}{dx} \to 0,\ \frac{dh_2}{dx} \to 0,\ \frac{dS}{dx} \to 0;$$

with $x \to +\infty$

$$u \to u_1,\ \rho \to \rho_1,\ h_2 \to -\frac{1}{3}\ln(\rho_1/\rho_0),\ S \to S_1,\ \frac{du}{dx} \to 0,$$
$$\frac{d\rho}{dx} \to 0,\ \frac{dh_2}{dx} \to 0,\ \frac{dS}{dx} \to 0.$$

Solutions of the system (2) were investigated in [4], where it was established that (2) has a continuous solution if u_0 is less than the rate of the propagation of longitudinal waves in an unstressed medium. Such solutions are called plastic waves; they will be discussed below.

Let the solution of system (1) $u_i(t, x_\alpha)$, $g_{ij}(t, x_\alpha)$, $S(t, x_\alpha)$ be known. The system of equations for the propagation of perturbations is obtained by the substitution of the perturbed solutions $u_i + \delta u_i$, $g_{ij} + \delta g_{ij}$, $S + \delta S$ into (1) and by discarding terms of the equations with perturbations with an order higher than the first. Carrying out this procedure, we obtain the linear system

$$\begin{gathered}
\frac{\partial \delta u_i}{\partial t} + u_\alpha \frac{\partial \delta u_i}{\partial x_\alpha} - c_{ikmn}\frac{\partial \delta g_{mn}}{\partial x_k} - c_{ik0}\frac{\partial \delta S}{\partial x_k} + \frac{\partial u_i}{\partial x_\alpha}\delta u_\alpha - \\
-\left(\frac{\partial c_{ikmn}}{\partial g_{pq}}\frac{\partial g_{mn}}{\partial x_k} + \frac{\partial c_{ik0}}{\partial g_{pq}}\frac{\partial S}{\partial x_k}\right)\delta g_{pq} - \left(\frac{\partial c_{ikmn}}{\partial S}\frac{\partial g_{mn}}{\partial x_k} + \frac{\partial c_{ik0}}{\partial S}\frac{\partial S}{\partial x_k}\right)\delta S = 0, \\
\frac{\partial \delta g_{ij}}{\partial t} + u_\alpha \frac{\partial \delta g_{ij}}{\partial x_\alpha} + g_{i\alpha}\frac{\partial \delta u_\alpha}{\partial x_j} + g_{j\alpha}\frac{\partial \delta u_\alpha}{\partial x_i} + \frac{\partial g_{ij}}{\partial x_\alpha}\delta u_\alpha + \\
+\frac{\partial u_\alpha}{\partial x_j}\delta g_{i\alpha} + \frac{\partial u_\alpha}{\partial x_i}\delta g_{j\alpha} = \frac{\partial \varphi_{ij}}{\partial g_{pq}}\delta g_{pq} + \frac{\partial \varphi_{ij}}{\partial S}\delta S, \\
\frac{\partial \delta S}{\partial t} + u_\alpha \frac{\partial \delta S}{\partial x_\alpha} + \frac{\partial S}{\partial x_\alpha}\delta u_\alpha = \frac{\partial \varkappa}{\partial g_{pq}}\delta g_{pq} + \frac{\partial \varkappa}{\partial S}\delta S,
\end{gathered} \tag{3}$$

where $c_{ikmn} = \frac{1}{\rho}\frac{\partial \sigma_{ik}}{\partial g_{mn}}$; $c_{ik0} = \frac{1}{\rho}\frac{\partial \sigma_{ik}}{\partial S}$. The coefficients of this system depend on the solution of (2), i.e., only on x.

We postulate that the perturbation of the front of the wave is "almost flat," i.e., that the derivatives with respect to y and z are much smaller than the derivatives with respect to x. Before passing on to the study of an arbitrarily perturbed front, we make an investigation for one harmonic of an expansion of the perturbation in a Fourier series (it is assumed that such an expansion is possible). Let the perturbation not depend on z, while the dependence on y is represented in the form $\delta u_i = \delta u_i(t, x)e^{i\omega y}$, $\delta g_{ij} = \delta g_{ij}(t, x)e^{i\omega y}$, $\delta S = \delta S(t, x)e^{i\omega y}$. For such perturbations we obtain a one-dimensional system of equations (we set $\delta u_3 = 0$, $\delta g_{13} = \delta g_{23} = 0$)

$$\begin{gathered}
\frac{\partial \delta u}{\partial t} + u\frac{\partial \delta u}{\partial x} - (E_{h_1h_1} - E_{h_1})\frac{\partial \delta h_1}{\partial x} - (E_{h_1h_2} - E_{h_1})\frac{\partial \delta h_2}{\partial x} - (E_{h_1h_3} - E_{h_1})\times \\
\times\frac{\partial \delta h_3}{\partial x} - E_{h_1S}\frac{\partial \delta S}{\partial x} + \frac{du}{dx}\delta u - \frac{1}{\rho}\frac{d\rho E_{h_1h_1}}{dx}\delta h_1 - \frac{1}{\rho}\frac{d\rho E_{h_1h_2}}{dx}\delta h_2 - \\
-\frac{1}{\rho}\frac{d\rho E_{h_1h_3}}{dx}\delta h_3 - \frac{1}{\rho}\frac{d\rho E_{h_1S}}{dx}\delta S - i\omega\frac{E_{h_1} - E_{h_2}}{g_1 - g_2}\delta g_{12} = 0, \\
\frac{\partial \delta h_1}{\partial t} + u\frac{\partial \delta h_1}{\partial x} - \frac{\partial \delta u}{\partial x} + \frac{dh_1}{dx}\delta u = -\frac{\partial \psi_1}{\partial h_i}\delta h_i - \frac{\partial \psi_1}{\partial S}\delta S, \\
\frac{\partial \delta h_2}{\partial t} + u\frac{\partial \delta h_2}{\partial x} + \frac{dh_2}{dx}\delta u - i\omega\delta v = -\frac{\partial \psi_2}{\partial h_i}\delta h_i - \frac{\partial \psi_2}{\partial S}\delta S,
\end{gathered}$$

$$\frac{\partial \delta h_3}{\partial t} + u\frac{\partial \delta h_3}{\partial x} + \frac{dh_3}{dx}\delta u = -\frac{\partial \psi_3}{\partial h_i}\delta h_i - \frac{\partial \psi_3}{\partial S}\delta S,$$

$$\frac{\partial \delta S}{\partial t} + u\frac{\partial \delta S}{\partial x} + \frac{dS}{dx}\delta u = \frac{\partial \varkappa}{\partial h_i}\delta h_i + \frac{\partial \varkappa}{\partial S}\delta S,$$

$$\frac{\partial \delta v}{\partial t} + u\frac{\partial \delta v}{\partial x} - \frac{E_{h_1} - E_{h_2}}{g_1 - g_2}\frac{\partial \delta g_{12}}{\partial x} - \frac{1}{\rho}\frac{d}{dx}\left(\rho\frac{E_{h_1} - E_{h_2}}{g_1 - g_2}\right)\delta g_{12} -$$

$$- i\omega(E_{h_1h_2} - E_{h_2})\delta h_1 - i\omega(E_{h_2h_2} - E_{h_2})\delta h_2 - i\omega(E_{h_2h_3} - E_{h_2})\delta h_3 - i\omega E_{h_2S}\delta S = 0,$$

$$\frac{\partial \delta g_{12}}{\partial t} + u\frac{\partial \delta g_{12}}{\partial x} + g_2\frac{\partial \delta v}{\partial x} + \frac{du}{dx}\delta g_{12} + i\omega g_1\delta u + \frac{1}{\tau}\delta g_{12} = 0,$$

where $\psi_i = \frac{1}{\tau}\left(h_i - \frac{h_1 + h_2 + h_3}{3}\right)$; $\varkappa = \frac{E_{h_i}\psi_i}{E_S}$; $\delta h_i = -\frac{1}{2g_i}\delta g_i$. This system can be brought into canonical form

$$\frac{\partial}{\partial t}\begin{pmatrix} Z \\ Y \end{pmatrix} + \begin{pmatrix} D^1 & 0 \\ 0 & D^2 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} Z \\ Y \end{pmatrix} + \begin{pmatrix} A^1 & 0 \\ 0 & A^2 \end{pmatrix}\begin{pmatrix} Z \\ Y \end{pmatrix} = \omega\begin{pmatrix} 0 & P^1 \\ P^2 & 0 \end{pmatrix}\begin{pmatrix} Z \\ Y \end{pmatrix},$$

where Z and Y are vectors of dimensionality 5 and 2, respectively; the matrices D^1 (5 × 5) and D^2 (2 × 2) are diagonal; the matrices D^1, D^2, A^1 (5 × 5), A^2 (2 × 2), P^1 (5 × 2), P^2 (2 × 5) depend on x [more exactly, on the solution of (2)]. The components of the vectors Z and Y are expressed in terms of δu, δv, δh_i, δg_{12}, δS as follows:

$$Z_1 = \delta u + \frac{1}{\rho c}\delta\sigma_1,\quad Z_2 = \delta u - \frac{1}{\rho c}\delta\sigma_1,\quad Z_3 = \delta h_2,\quad Z_4 = \delta h_3,\quad Z_5 = \delta S,$$
$$Y_1 = i\left(\delta v + \frac{1}{g_2}b\delta g_{12}\right),\quad Y_2 = i\left(\delta v - \frac{1}{g_2}b\delta g_{12}\right), \tag{4}$$

where $c = \sqrt{E_{h_1h_1} - E_{h_1}}$ is the rate of propagation of the longitudinal waves; $b = \sqrt{g_2\frac{E_{h_1} - E_{h_2}}{g_2 - g_1}}$ is the rate of propagation of the transverse waves. The nonnull elements of the matrices D^1 and D^2 are $D^1_{11} = u - c$, $D^1_{22} = u + c$, $D^1_{33} = D^1_{44} = D^1_{55} = u$, $D^2_{11} = u - b$, $D^2_{22} = u + b$. All the matrices D^i, A^i, and P^i are real.

We shall seek the solution in the form $Z_i = Z_i(x)e^{\lambda t}$, $Y_i = Y_i(x)e^{\lambda t}$; we obtain a problem for the eigenvalues

$$\lambda\begin{pmatrix} Z \\ Y \end{pmatrix} + \begin{pmatrix} D^1 & 0 \\ 0 & D^2 \end{pmatrix}\frac{d}{dx}\begin{pmatrix} Z \\ Y \end{pmatrix} + \begin{pmatrix} A^1 & 0 \\ 0 & A^2 \end{pmatrix}\begin{pmatrix} Z \\ Y \end{pmatrix} = \omega\begin{pmatrix} 0 & P^1 \\ P^2 & 0 \end{pmatrix}\begin{pmatrix} Z \\ Y \end{pmatrix} \tag{5}$$

with these boundary conditions: for $x \to -\infty$, $Z_2, Z_3, Z_4, Z_5, Y_1, Y_2 \to 0$; with $x \to +\infty$, $Z_1 \to 0$.

We shall consider the solution of system (4) with small values of ω, i.e., with waves of perturbation with respect to y, whose length is great in comparison with the thickness of the front of the plastic wave, and shall seek the solution in the form of an asymptotic series in powers of ω in the vicinity of $\omega = 0$.

It is found that, with $\omega = 0$, the system (5) has $\lambda = 0$ as an eigenvalue. The eigenfunction corresponding to $\lambda = 0$ can be obtained from Eqs. (2), describing the structure of the plastic wave. If, for system (2), we write the equations for perturbations, we then obtain the system

$$\delta\rho = -\rho(\delta h_1 + 2\delta h_2) = 2\rho\frac{E_{h_1h_1} - E_{h_1h_2} - (E_{h_1} - E_{h_2})\frac{E_{h_1S}}{E_S}}{u^2 - c^2}\delta h_2,$$
$$\delta u = -\frac{u}{\rho}\delta\rho,\quad \delta S = 2\frac{E_{h_1} - E_{h_2}}{E_S}\delta h_2, \tag{6}$$

$$u\frac{d\delta h_2}{dx}=\frac{2}{3\tau}\left\{\left(h_1-h_2-1+\frac{h_1-h_2}{\tau}\tau_{h_1}\right)\frac{E_{h_1h_1}-E_{h_1h_2}-(E_{h_1}-E_{h_2})\cdot\frac{E_{h_1S}}{E_S}}{u^2-c^2}-\right.$$

$$\left.-\frac{3}{2}+(h_1-h_2)\frac{\tau_{h_1}-\tau_{h_2}}{\tau}-(h_1-h_2)(E_{h_1}-E_{h_2})\frac{\tau_S}{E_S\tau}\right\}\delta h_2.$$

The eigenfunction for $\lambda = 0$ is a nontrivial solution of this system and represents the difference (in a linear approximation) between the given wave and the same wave shifted in a parallel manner along the x axis [the shifted wave satisfies (2) by virtue of the arbitrariness in the choice of the origin of coordinates $x = 0$ with the search for the solution of (2)]. For the solution of (6), $\delta\rho$, δh_2, δS, $\delta u \to 0$ with $x \to \pm\infty$. Thus, with $\omega = 0$, the system (5) has $\lambda = 0$ as an eigenvalue and the eigenfunction $Y^0 = 0$, Z^0 [Z^0 is obtained by solution of (6)]; here $Z_i^0 \to 0$ with $x \to \pm\infty$.

We shall seek the solution of (5) in the form $Z = Z^0 + \omega Z^1 + \omega^2 Z^2 + \ldots$, $Y = Y^0 + \omega Y^1 + \omega^2 Y^2 + \ldots$, $\lambda = \lambda^0 + \omega\lambda^1 + \omega^2\lambda^2 + \ldots$.

It is well known that $\lambda^0 = 0$, and that $Y^0 = 0$ and Z^0 are known [obtained by solution of (6)]. We shall assume that $\lambda^1 = 0$, $\lambda^2 = 0$. Setting $\lambda = 0$ in (5), and substituting here the expansions of Z and Y, we obtain the resolvent for Z^i and Y^i

$$D^1\frac{dZ^1}{dx}+A^1Z^1=P^1Y^0=0;\quad D^2\frac{dY^1}{dx}+A^2Y^1=P^2Z^0. \tag{7}$$

The equation for Z^1 coincides with the equation for Z^0 and, since $Z^1 = Z^0$ does not contain new information on the behavior of the perturbations, it can be assumed that $Z^1 = 0$; we find Y^1 from (7) as the integral of an equation with the known right-hand part P^2Z^0. For Z^2, Y^2 we have

$$D^1\frac{dZ^2}{dx}+A^1Z^2=P^1Y^1;\quad D^2\frac{dY^2}{dx}+A^2Y^2=P^2Z^1=0. \tag{8}$$

From this it can be seen that $Y^2 = 0$, while Z^2 is found from (8) as the integral of an equation with a known right-hand part P^1Y^1. The following terms of the expansions of Z and Y can be found in exactly the same way.

Thus, we have $Z = Z^0 + \omega^2Z^2 + O(\omega^4)$, $Y = \omega Y^1 + O(\omega^3)$. Consequently, a perturbation of the front of the wave of the form $e^{i\omega y}$ leads to the formation of shear waves of the perturbations on the order of ω and to the appearance of longitudinal waves of the perturbations, which behind the wave lead to the redistribution of the principal stresses by an amount on the order of ω^2.

We now pass on to an investigation of an arbitrarily perturbed front of a wave; we shall seek steady-state perturbations. In system (3), we discard derivatives with respect to t. It is well known that a system obtained in this manner has a solution of the shifts of the wave along the x axis (6). We postulate that this shift, at each point, is due to a change in the surface of the front of the wave, which is given by the function $\xi(y, z)$. We shall assume that

$$\xi \gg \xi_y \gg \xi_{yy} \gg \ldots,\ \xi \gg \xi_z \gg \xi_{zz} \gg \ldots. \tag{9}$$

Thus, we shall seek the solution of a steady-state system for perturbations of the following form:

$$\begin{aligned}\delta u_i &= \delta u_i^0(x)\,\xi(y,z)+\delta u_i^1(x,y,z)+\delta u_i^2(x,u,z)+\ldots,\\ \delta g_{ij} &= \delta g_{ij}^0(x)\,\xi(y,z)+\delta g_{ij}^1(x,y,z)+\delta g_{ij}^2(x,y,z)+\ldots,\\ \delta S &= \delta S^0(x)\,\xi(y,z)+\delta S^1(x,y,z)+\delta S^2(x,y,z)+\ldots.\end{aligned} \tag{10}$$

We assume that, in these expansions, each succeeding term is much less than the preceding, and that $\partial\delta u^i/\partial x \sim \partial\delta u^{i-1}/\partial y$. We write again in more detail the steady-state system obtained from (3):

$$\rho u \frac{\partial \delta u}{\partial x} - \frac{\partial \delta \sigma_{11}}{\partial x} = \frac{\partial \delta \sigma_{12}}{\partial y} + \frac{\partial \delta \sigma_{13}}{\partial z},$$
$$\rho u \frac{\partial \delta v}{\partial x} - \frac{\partial \delta \sigma_{21}}{\partial x} = \frac{\partial \delta \sigma_{22}}{\partial y} + \frac{\partial \delta \sigma_{23}}{\partial z},$$
$$\rho u \frac{\partial \delta w}{\partial x} - \frac{\partial \delta \sigma_{31}}{\partial x} = \frac{\partial \delta \sigma_{32}}{\partial y} + \frac{\partial \delta \sigma_{33}}{\partial z},$$
$$u \frac{\partial \delta h_1}{\partial x} - \frac{\partial \delta u}{\partial x} + \frac{dh_1}{dx} \delta u + \delta \psi_1 (h_1, h_2, h_3, S) = 0,$$
$$u \frac{\partial \delta h_2}{\partial x} + \frac{dh_2}{dx} \delta u + \delta \psi_2 (h_1, h_2, h_3, S) = \frac{\partial \delta v}{\partial y}, \qquad (11)$$
$$u \frac{\partial \delta h_3}{\partial x} + \frac{dh_3}{dx} \delta u + \delta \psi_3 (h_1, h_2, h_3, S) = \frac{\partial \delta w}{\partial z},$$
$$u \frac{\partial \delta g_{12}}{\partial x} + g_2 \frac{\partial \delta v}{\partial x} + \frac{du}{dx} \delta g_{12} + \frac{\delta g_{12}}{\tau} = -g_1 \frac{\partial \delta u}{\partial y},$$
$$u \frac{\partial \delta g_{13}}{\partial x} + g_3 \frac{\partial \delta w}{\partial x} + \frac{du}{dx} \delta g_{13} + \frac{\delta g_{13}}{\tau} = -g_1 \frac{\partial \delta u}{\partial z},$$
$$u \frac{\partial \delta g_{23}}{\partial x} + \frac{\delta g_{23}}{\tau} = -g_2 \frac{\partial \delta v}{\partial z} - g_3 \frac{\partial \delta w}{\partial y},$$
$$u \frac{\partial \delta S}{\partial x} + \frac{dS}{dx} \delta u - \delta \varkappa (h_1, h_2, h_3, S) = 0.$$

We substitute the expansion (10) into (11). We obtain equations for δu_i^0, δh_i^0, δg_{ij}^0, δS^0 by discarding all the remaining terms of the expansion and [taking account of (9)] the derivatives of δu_i^0, δh_i^0, δg_{ij}^0, δS^0 with respect to y and z.

The solutions of the equations for δu_i^0, δh_i^0, δg_{ij}^0, δS^0 are solutions of the system (6). They can be found in the form $\delta u^0(x) = du/dx$, $\delta h_i^0(x) = dh_i/dx$, $\delta S^0(x) = dS/dx$, $\delta g_{ij}^0 = 0$, $\delta v^0 = \delta w^0 = 0$, where u, h_i, S is the solution of system (2).

For δu^1, δh_i^1, δS^1 we have the same system as for zero approximations, since $\delta g_{ij}^0 = 0$ and $\delta \sigma_{ij}^0 = 0$ $(i \neq j)$, and we assume that $\delta u^1 = 0$, $\delta h_i^1 = 0$, $\delta S^1 = 0$.

For δv^1, δg_{12}^1, we have

$$\rho u \frac{\partial \delta v^1}{\partial x} - \frac{\partial \delta \sigma_{21}^1}{\partial x} = \delta \sigma_{22}^0 \xi_y (y, z),$$
$$u \frac{\partial \delta g_{12}^1}{\partial x} + g_2 \frac{\partial \delta v^1}{\partial x} + \frac{du}{dx} \delta g_{12}^1 + \frac{1}{\tau} \delta g_{12}^1 = -g_1 \delta u^0 \xi_y (y, z).$$

Assuming that $\delta v^1 = \delta v^1(x)\xi_y(y, z)$, $\delta g_{12}^1 = \delta g_{12}^1(x)\xi_y(y, z)$, for $\delta v^1(x)$, δg_{12}^1, we obtain a system of ordinary equations, leading to the system (7), which has been considered in the investigation of harmonic perturbations. Further, assuming that $\delta w^1 = \delta w^1(x)\xi_z(y, z)$, $\delta g_{13}^1 = \delta g_{13}^1(x)\xi_z(y, z)$, for $\delta w^1(x)$, $\delta g_{13}^1(x)$, we obtain exactly the same systems as for $g_2 = g_3$, $\partial\sigma_{21}/\partial g_{12} = \partial\sigma_{31}/\partial g_{13}$, and $\delta\sigma_{22}^0 = \delta\sigma_{33}^0$.

Consequently, $\delta v^1(x) = \delta w^1(x)$, $\delta g_{12}^1(x) = \delta g_{13}^1(x)$.

And, for δg_{23}^1, we obtain

$$u \frac{\partial \delta g_{23}^1}{\partial x} + \frac{1}{\tau} \delta g_{23}^1 = -g_2 \delta v^0 \xi_z (y, z) - g_3 \delta w^0 \xi_y (y, z),$$

since $\delta v^0 = 0$, $\delta w^0 = 0$; i.e., $\delta g_{23}^1 = 0$.

We now examine the second terms of the expansion (10). For δu^2, δh_i^2, δS^2 we obtain

$$\rho u \frac{\partial \delta u^2}{\partial x} - \frac{\partial \delta \sigma_{11}^2}{\partial x} = \delta \sigma_{12}^1 (x) \xi_{yy} (y, z) + \delta \sigma_{13}^1 (x) \xi_{zz} (y, z),$$
$$u \frac{\partial \delta h_1^2}{\partial x} - \frac{\partial \delta u^2}{\partial x} + \frac{dh_1}{dx} \delta u^2 + \frac{\partial \psi_1}{\partial h_i} \delta h_i^2 + \frac{\partial \psi_1}{\partial S} \delta S^2 = 0,$$

$$u\frac{\partial \delta h_2^2}{\partial x} + \frac{dh_2}{dx}\delta u^2 + \frac{\partial \psi_2}{\partial h_i}\delta h_i^2 + \frac{\partial \psi_2}{\partial S}\delta S^2 = \delta v^1(x)\xi_{yy}(y, z),$$

$$u\frac{\partial \delta h_3^2}{\partial x} + \frac{dh_3}{dx}\delta u^2 + \frac{\partial \psi_3}{\partial h_i}\delta h_i^2 + \frac{\partial \psi_3}{\partial S}\delta S^2 = \delta w^1(x)\xi_{zz}(y, z),$$

$$u\frac{\partial \delta S^2}{\partial x} + \frac{dS}{dx}\delta u^2 - \frac{\partial \varkappa}{\partial h_i}\delta h_i^2 - \frac{\partial \varkappa}{\partial S}\delta S^3 = 0.$$

We shall seek the solution of this system in the form $\delta u^2 = \delta u^{21}(x)\xi_{yy} + \delta u^{22}(x)\xi_{zz}$ (the remaining sought functions are represented in the same way). For terms proportional to ξ_{yy}, we obtain a system reducing to (8), and for terms proportional to ξ_{zz} we obtain the same system if δh_2 and δh_3 change places. Thus, we obtain the solutions

$$\delta u^2 = \delta u^2(x)(\xi_{yy} + \xi_{zz}), \quad \delta h_1^2 = \delta h_1^2(x)(\xi_{yy} + \xi_{zz}),$$

$$\delta S^2 = \delta S^2(x)(\xi_{yy} + \xi_{zz}),$$

$$\delta h_2^2 = \delta h_2^2(x)\xi_{yy} + \delta h_3^2(x)\xi_{zz}, \quad \delta h_3^2 = \delta h_3^2(x)\xi_{yy} + \delta h_2^2(x)\xi_{zz},$$

where $\delta u^2(x)$, $\delta h_i^2(x)$, $\delta S^2(x)$, $\delta h_2^2(x)$, $\delta h_3^2(x)$ can be found from solutions of system (8), which is written with the investigation of harmonic functions.

For δv^2, δg_{12}^2, δw^2, δg_{13}^2 the same equations are obtained as for the first terms of the expansion; therefore, it can be assumed that $\delta v^2 = \delta w^2 = \delta g_{12}^2 = \delta g_{13}^2 = 0$. And, for δg_{23}^2, the following equation is obtained:

$$u\frac{\partial \delta g_{23}^2}{\partial x} + \frac{1}{\tau}\delta g_{23}^2 = -g_2\delta v^1(x)\xi_{yz} - g_3\delta w^1(x)\xi_{yz} = -2g_2\delta v^1(x)\xi_{yz}.$$

Thus, it is established that the steady-state system for the perturbations (11) has a solution of the following form:

$$\delta u = \delta u^0(x)\xi(y, z) + \delta u^2(x)(\xi_{yy} + \xi_{zz}) + \ldots,$$
$$\delta h_1 = \delta h_1^0(x)\xi(y, z) + \delta h_1^2(x)(\xi_{yy} + \xi_{zz}) + \ldots,$$
$$\delta S = \delta S^0(x)\xi(y, z) + \delta S^2(x)(\xi_{yy} + \xi_{zz}) + \ldots,$$
$$\delta h_2 = \delta h_2^0(x)\xi(y, z) + \delta h_2^2(x)\xi_{yy} + \delta h_3^2(x)\xi_{zz} + \ldots,$$
$$\delta h_3 = \delta h_2^0(x)\xi(y, z) + \delta h_3^2(x)\xi_{yy} + \delta h_2^2(x)\xi_{zz} + \ldots,$$
$$\delta v = \delta v^1(x)\xi_y + \ldots, \quad \delta g_{12} = \delta g_{12}^1(x)\xi_y + \ldots,$$
$$\delta w = \delta v^1(x)\xi_z + \ldots, \quad \delta g_{13} = \delta g_{12}^1(x)\xi_z + \ldots,$$
$$\delta g_{23} = \delta g_{23}^2(x)\xi_{yz} + \ldots.$$

From these formulas it can be seen that a shift of the front of the wave, due to a perturbation of its surface $\xi(y, z)$, leads to the appearance of transverse wave perturbations proportional to $\xi_y(y, z)$ and $\xi_z(y, z)$, and these lead to the appearance, beyond the perturbed section of the front, of principal deformations of the form

$$\delta h_1^2(x)(\xi_{yy} + \xi_{zz}), \quad \delta h_2^2(x)\xi_{yy} + \delta h_3^2(x)\xi_{zz}, \quad \delta h_3^2(x)\xi_{yy} + \delta h_2^2(x)\xi_{zz}.$$

These deformations obviously leave a trace in the metal after the passage of the wave, which was observed in the experiments described in [5].

In conclusion, we give some values of δh_i^2, obtained by numerical calculations. For copper (equation of state from [3], relaxation time from [2]), immediately behind the wave with $\tau = 10^{-4}$ sec we have the following:

compression behind wave $\rho_1/\rho_0 = 1.11$

$$\delta h_1^2 = 0.2889, \quad \delta h_2^2 = 0.1853, \quad \delta h_3^2 = 0.1854;$$

compression behind wave $\rho_1/\rho_0 = 1.09$

$$\delta h_1^2 = 0.2953, \quad \delta h_2^2 = 0.2095, \quad \delta h_3^2 = 0.2096;$$

compression behind wave $\rho_1/\rho_0 = 1.05$

$$\delta h_1^2 = 0.3184, \quad \delta h_2^2 = 0.2216, \quad \delta h_3^2 = 0.2217.$$

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EQUATIONS OF THE LINEAR THEORY OF ELASTICITY WITH POINT MAXWELLIAN SOURCES OF STRESS RELAXATION

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1. General Solution. Relationships on the Characteristic

The system being studied has the form

$$\begin{aligned}
&\rho_0 U\frac{\partial u}{\partial x} - \frac{\partial \sigma_{11}}{\partial x} - \frac{\partial \sigma_{12}}{\partial y} = 0,\\
&\rho_0 U\frac{\partial v}{\partial x} - \frac{\partial \sigma_{12}}{\partial x} - \frac{\partial \sigma_{22}}{\partial y} = 0,\\
&U\frac{\partial \sigma_{11}}{\partial x} - \rho_0 c_0^2\frac{\partial u}{\partial x} - \rho_0\left(c_0^2 - 2b_0^2\right)\frac{\partial v}{\partial y} = 0,\\
&U\frac{\partial \sigma_{22}}{\partial x} - \rho_0\left(c_0^2 - 2b_0^2\right)\frac{\partial u}{\partial x} - \rho_0 c_0^2\frac{\partial v}{\partial y} = 0,\\
&U\frac{\partial \sigma_{33}}{\partial x} - \rho_0\left(c_0^2 - 2b_0^2\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0,\\
&U\frac{\partial \sigma_{12}}{\partial x} - \rho_0 b_0^2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0,
\end{aligned} \tag{1.1}$$

where σ_{11}, σ_{22}, σ_{33}, σ_{12} are the components of the stress tensor; u + U is the horizontal component of the vector of the displacement rate of points of the medium ($U < b_0 < c_0 \ll U$); v is the vertical component of the vector of the displacement rate of points of the medi-

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